

On the equation of motion for test particles in an ambient gravitational field as the Wong equation for a generalized gauge theory

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Abstract

In this note we demonstrate the equation of motion for test particles in an ambient gravitational field for the teleparallel theory of gravity, considered as a generalized gauge theory, using a computational scheme due to Feynman. It can be thought of as the Wong equation for a generalized gauge theory. The Wong and Lorentz equations become identical when the generators of a generalized non-abelian gauge theory are taken to be the local translation generators.

1 Introduction

The study of the mathematical and conceptual relationship between Yang-Mills gauge theory and Einstein's general relativity has occupied a central place in the efforts of theoretical physicists for some time now. From one point of view it is the problem of the unification of forces, since three quarters of Nature is described well by gauge theory and the other quarter is described well by the classical theory of general relativity. From another point of view it is a fundamental issue in the search for a consistent quantum theory of gravity. Gauge theoretic ideas are not only consistent with quantum theoretic ones, but may even form an essential part of quantum theory, for example taming the theory when we aspire to quantize many degrees of freedom. Even more fundamentally, gauge systems appear immediately from quantum dynamics in the form of Berry's connection [2] and the non-abelian connections studied by Wilczek and Zee [27].

The most precise relationships between gauge theory and gravity are, in our opinion, the various relationships uncovered within the scheme of string theory and the results on the level of the classical actions initiated by Utiyama [25] and discussed by many authors. An excellent bibliography of this literature is provided in the book [3]. These two approaches have one radical difference: in

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the former the association between gauge and gravity observables is non-local, whereas in the latter it has been assumed local—indeed gravity is conceived of as a local gauge theory. We think that this difference is due to both the fact that the string theory approach includes quantum corrections while the alternative approach does not, and because, even on the classical level the approach initiated by Utiyama is not fully developed conceptually or mathematically. In light of the direction that the latter approach is taking, it seems very likely that this difference will evaporate.

In this note we calculate the equation of motion for test particles in an ambient gravitational field, in a way that makes it clear that it is a Wong equation for a generalized gauge theory. We say *generalized gauge theory* because the concept of local gauge symmetry, as it appears in Yang-Mills theory, has to be extended to include situations in which the structure constants are functions on space-time. In this sense the symmetry itself is dynamically determined—although we do not introduce the dynamics for it here. The dynamics of the symmetry corresponds to the kinetic terms for the gravitational field. In order to obtain the equations of motion for test particles in an ambient gravitational field we extend a calculational scheme due to Feynman to incorporate non-Abelian Yang-Mills fields. This was started in [22] although some results are missing, among them the relation between the gauge fields and field strength and the transformation law for gauge fields. These results and some others are obtained before we treat the gravitational case.

The final result can be stated very simply. The Wong equation for Yang-Mills theory is

$$\dot{I}^a - f^{ab}{}_c A_{b\mu} \dot{x}^\mu I^c = 0. \quad (1)$$

By the trivial replacements

$$I^a \mapsto \mathcal{P}^a(x) \quad (2)$$

$$f^{ab}{}_c \mapsto T^{ab}{}_c(x) \quad (3)$$

$$A_{b\mu}(x) \mapsto h_{b\mu}(x) \quad (4)$$

this equation becomes the Wong equation *and* the Lorentz equation for the teleparallel equivalent of general relativity¹, and is therefore further equivalent to the geodesic equation. To the right of the arrows we have the a generalized momentum, the torsion and the frame components respectively. Note carefully that gravity is both a specialization and a generalization of ordinary gauge theory: It is a special case due to the degeneracy of the Lorentz and Wong equations of the relevant generalized gauge theory, and a generalization due to the space-time dependence of what were formerly the structure constants. Note also how that characteristic quadratic velocity dependence of the geodesic equation arises: one velocity already appears in the ordinary Wong equation while the other is present because of the nature of the generator $\mathcal{P}^a(x, p)$.

¹For a review of the teleparallel equivalent of general relativity, see [1]

2 Gauge Theory in Feynman's scheme

Here we review the treatment of gauge theories with gauge groups that are the direct sum of commuting compact simple and $U(1)$ subalgebras [26], following [18], [22] with some refinements. As the results are the *non-abelian analogue of the Lorentz force law*, the *Wong equations* for the gauge generators, the *Bianchi identities* for the field strengths, the *relation between the gauge field and gauge field strengths*, and the *transformation law for the gauge fields*, we comment here on the range of validity of these.² This happens because the symmetry considerations adhered to do not forbid the appearance of fundamental scalars. The resulting theory is only relevant in the limit in which matter field excitations can be treated as point particles while the Yang-Mills field is governed by classical wave equations. This can only happen at extremely high energies due to the mass gap, see e.g. [28] for a review. In simple terms, the Yang-Mills-matter system is treated in precise analogy with the classical electrodynamics of point charges; of course, as already stated, the physical conditions under which this approximation is valid may be exotic.

As a starting point we consider the following fundamental assumptions as the basis of the computational scheme employed ³:

- A particle moves along a trajectory $x^\mu(\tau)$ in $\mathbb{R}^{3,1}$, and it possess the physical properties **mass** and **color**, represented by m and $I^a(\tau)$ respectively. $I^a(\tau_0)$ is in the Lie algebra \mathfrak{g} of the gauge group G for fixed $\tau_0 \in \mathbb{R}$.⁴
- The operators $\{x^\mu(\tau), \dot{x}^\mu(\tau), I^a(\tau)\}$ satisfy the following **Yang-Mills-Feynman algebra** (which we will call \mathfrak{J}) of **Feynman brackets** (**F-brackets**):

$$[x^\mu, x^\nu] = 0 \quad (5)$$

$$[x^\mu, \dot{x}^\nu] = \left(\frac{-i\hbar}{m}\right) \eta^{\mu\nu} \quad (6)$$

$$[I^a, I^b] = i\hbar f_c^{ab} I^c \quad (7)$$

$$[x^\mu, I^a] = 0 \quad (8)$$

These brackets satisfy **all the properties of standard commutators resp. Poisson brackets**, in addition to a further Leibniz rule we call the

²The result contains a scalar field contribution to the equations of motion as well as the usual Lorentz term. This term appears with its own field strength and Bianchi identity.

³ $x^\mu(\tau)$ is a Heisenberg picture operator while, when the τ disappears, we are discussing the Schrödinger picture operator. Thus when interpreting an object like x^μ we do not merely indicate a point on the trajectory, but a full fledged coordinate on which a field may depend, as in $A_\mu(x)$. This is part of a deeper set of issues, falling under the rubrics *restriction to one particle operators* and *stability*, which will be addressed in future work.

⁴There is some confusion in [22] surrounding the association of the parameter τ with the proper time which arises from an incorrect handling of constraints (see [11] page 19) in the Poisson bracket formalism. This will be dealt with in connection with the stability issue alluded to above, which arises from the indefinite metric η in the algebra. Remarkably, Grassmann variables and supersymmetry are forced on our formalism when we try to make a Lagrangian for the Wong equations; this will be discussed in future work.

Lie-Leibniz rule⁵ (since it enforces the Liebniz rule with respect to the Lie derivative).

$$\mathcal{L}_{\partial_\tau}[A, B] = [\mathcal{L}_{\partial_\tau} A, B] + [A, \mathcal{L}_{\partial_\tau} B]. \quad (9)$$

Direct products of gauge algebras are considered later, for now we imagine $\{I^a\}$ generates a gauge algebra \mathfrak{g} .

- The coordinate operators obey second order differential equations of motion, i.e. the Heisenberg equations of motion in the form of **Lorentz's law**,

$$m\ddot{x}^\mu = \mathcal{F}^\mu(x, \dot{x}, I), \quad (10)$$

where the force is assumed to be *linear* in the $\{I^a\}$.

- The color variables are assumed to satisfy **first order differential equations of motion**, as per usual:

$$\dot{I}^a = \mathcal{H}^a(x, \dot{x}, I). \quad (11)$$

In the non-Abelian case these are called **Wong's equations**. Again, $\mathcal{H}^a(x, \dot{x}, I)$ is assumed *linear*. The reason why internal variables satisfy first order equations of motion while variables associated with the base space (space-time) satisfy second order equations is simply because in the latter case the generator is the momentum, which can *itself* be written as a time derivative. Thus truly, all the equations of motion are first order equations for the relevant generators as is obvious in Hamilton's framework. In the gravitational case, i.e. when the generator is taken to be the local translation generator and a generalized local gauge theory is constructed, the Lorentz equations and the Wong equations coincide.

Feynman's original motivation was to create a formalism robust enough to encompass systems that are not Lagrangian or Hamiltonian. Our motivation for adopting his approach is completely different, as we simply wish to show that gravity's fundamental *generalized gauge origin* is rather clear in his language; similarities and differences between the gravitational and Yang-Mills cases of gauge theory⁶ are put in stark relief. This is just as well; since Dyson's publication of Feynman's proof no one has succeeded in utilizing the formalism for

⁵The implication of the assumed rule (9) is significant and was not appreciated in the pioneering literature. It can be shown that a dynamical system is Hamiltonian if and only if the time derivative acts on Poisson brackets in accordance with (9), thus stopping Feynman's initial motivation in its tracks. For a proof see e.g. [13]. This explains why exhaustive searches for novel (non-Hamiltonian) systems within the Feynman scheme, as e.g. conducted in [6], turned up nothing.

⁶Here we use the term *generalized gauge theory* in a loose way to denote a theoretical structure that will encompass both local Yang-Mills symmetries and local translation symmetries, which can be thought of as at the root of gravity. We assume little more than that the symmetry itself is a dynamical variable through the introduction of space-time dependent structure "constants". It is pleasing that the generalization can be stated in simple terms and takes Weyl's definition of local symmetry one clear step further.

its original purpose because all authors have assumed the Lie-Leibniz rule (9). In fact one can show that (9) implies the existence of an Hamiltonian. The formalism has however been used to obtain results in the noncommutative case⁷.

Instead of postulating the Wong equation, as in [22], we *derive* it in complete analogy with the derivation of the Lorentz force using (11) and define the gauge potential through

$$\check{\mathcal{A}}^{\mu a}(x, \dot{x}, I) := \frac{m}{i\hbar}[\dot{x}^\mu, I^a]. \quad (12)$$

Surprisingly, the ordinary gauge potential is obscured in (12). The index a is a matrix index of \mathfrak{g} in the adjoint representation and *not* an index specifying a Lie algebra generator. Specifically, the usual form of the gauge potential is related to the one in (12) by

$$\check{\mathcal{A}}^{\mu a} = (\mathcal{A}^\mu{}_c)(\check{I}^c)^a{}_b I^b, \quad (13)$$

where the over-symbol in (13) designates the *adjoint representation* of \mathfrak{g} , explaining its appearance in (12) as well. It is easy to see that the Yang-Mills gauge potential does not depend on the velocity by taking the F-bracket of (12) with \dot{x}^ν and using the Jacobi identity. The linearity condition then implies

$$[\dot{x}^\mu, I^a] := \frac{i\hbar}{m}\check{\mathcal{A}}^{\mu a}{}_b(x)I^b. \quad (14)$$

Taking the F-bracket of (11) with x^μ and using (14) we find

$$[x^\mu, \mathcal{H}^a{}_b(x, \dot{x})]I^b = -\frac{i\hbar}{m}\check{\mathcal{A}}^{\mu a}{}_b(x)I^b. \quad (15)$$

Now we can use the linear independence of the generators and the fact that $[x_\nu, f(\dot{x})] = -\frac{i\hbar}{m}\partial_{\dot{x}^\nu}f(\dot{x})$; we find, after integration,

$$\dot{I}^a - \check{\mathcal{A}}_\mu{}^a{}_b(x)\dot{x}^\mu I^b = 0, \quad (16)$$

which is the *Wong equation*. It can be written, alternatively,

$$\dot{I}^a - f_c{}^{ab}A_{b\mu}\dot{x}^\mu I^c = 0, \quad (17)$$

using (13).

The *field strength* corresponding to the potential just introduced is

$$F_{\mu\nu}(x) := -\frac{m^2}{i\hbar}[\dot{x}_\mu, \dot{x}_\nu]. \quad (18)$$

The components of $F_{\mu\nu}$ in the Lie algebra basis $\{I^a\}$ can be worked out in the form

$$\begin{aligned} F^{\mu\nu} &= -\frac{m^2}{i\hbar}[\dot{x}^\mu, \dot{x}^\nu] \\ &= \frac{m}{i\hbar}[x^\mu, \mathcal{F}^\nu] \\ &= \frac{m}{i\hbar}[x^\mu, \mathcal{F}_a^\nu]I^a \end{aligned}$$

⁷See however [5]

implying

$$F_{\mu\nu}(x, I) = F_{a\mu\nu}(x)I^a. \quad (19)$$

That this is in fact related to \check{A}_μ in the usual way will be shown ⁸ shortly. Note that the a index in (19) indicates a particular basis element of \mathfrak{g} and has nothing to do with the adjoint representation.

First we can get the *Bianchi identity* for $F_{\mu\nu}$ out of the way since it reduces to the Jacobi identity for the velocity operators,

$$[\dot{x}_\mu, [\dot{x}_\nu, \dot{x}_\rho]] + \text{cyclic} \quad (20)$$

$$= \frac{i\hbar}{m^2} ([\dot{x}_\mu, F_{\nu\rho}] + \text{cyclic}) \quad (21)$$

$$= \frac{\hbar^2}{m^3} (\mathcal{D}_\mu F_{\nu\rho} + \text{cyclic}) = 0. \quad (22)$$

To get the last line we used the identity $[\dot{x}_\mu, \phi_a(x)I^a] = \frac{i\hbar}{m} (\mathcal{D}_\mu \phi)_a I^a$.

To prove that indeed the field strength introduced in (18) corresponds to the potential introduced in (12) and (13) we can start by taking the F-bracket of (14) with \dot{x}^ν

$$[\dot{x}_\nu, [\dot{x}_\mu, I^a]] + \left(\frac{i\hbar}{m}\right) [\dot{x}_\nu, \check{A}_\mu^a{}_b(x)I^b] = 0, \quad (23)$$

and then using Jacobi on the first term and Leibniz on the second we get

$$\begin{aligned} & [\dot{x}_\mu, [I^a, \dot{x}_\nu]] + [I^a, [\dot{x}_\nu, \dot{x}_\mu]] + \\ & \left(\frac{i\hbar}{m}\right) [\dot{x}_\nu, \check{A}_\mu^a{}_b(x)] I^b + \left(\frac{i\hbar}{m}\right) \check{A}_\mu^a{}_b [\dot{x}_\nu, I^b] = 0 \end{aligned} \quad (24)$$

leading to

$$\begin{aligned} & \left(\frac{-i\hbar}{m}\right) [\dot{x}_\mu, \check{A}_\nu^a{}_b I^b] - \left(\frac{i\hbar}{m^2}\right) [I^a, F_{\mu\nu}] + \\ & \left(\frac{i\hbar}{m}\right)^2 \check{A}_{\mu,\nu}^a{}_b I^b + \left(\frac{i\hbar}{m}\right)^2 \check{A}_\mu^a{}_b \check{A}_\nu^b{}_c I^c = 0 \end{aligned} \quad (25)$$

or⁹

$$-(\check{F}_{\mu\nu})^a{}_c I^c + 2(\check{A}_{[\mu,\nu]})^a{}_b I^b + 2\check{A}_{(\mu}^a{}_b \check{A}_{\nu)}^b{}_c I^c = 0. \quad (26)$$

Changing the dummy indices and using the linear independence of $\{I^a\}$ we have

$$(\check{F}_{\mu\nu})^a{}_b = 2(\check{A}_{[\mu,\nu]})^a{}_b + 2\check{A}_{(\mu}^a{}_c \check{A}_{\nu)}^c{}_b \quad (27)$$

which is the standard relation for the field strength in terms of the potential in Yang-Mills theory. It was simplest to prove in the adjoint representation but it

⁸The significance of the adjoint representation and the peculiar form of (12) were not appreciated in [22], which led to difficulty in the demonstration of this fact among others. Of course the result is representation independent.

⁹Notice the factors associated with the (anti)symmetrization of indices.

holds for any representation; we can simply use (13) and its analogue for $(\check{F}_{\mu\nu})^a_b$ in order to convert (27) into the representation independent form

$$F_{\mu\nu} = 2A_{[\mu,\nu]} + [A_\mu, A_\nu]. \quad (28)$$

In order to demonstrate the *Lorentz force* we introduce the function $\mathcal{G}^\mu(x, \dot{x}, I) = \mathcal{G}^\mu_a I^a$ through the following equation,

$$\mathcal{G}^\mu(x, I) := \mathcal{F}^\mu(x, \dot{x}, I) - F^{\mu\nu}(x, I)\dot{x}_\nu \quad (29)$$

where \mathcal{G}^μ_a can be shown¹⁰ to be a function of x only. A moments reflection shows that the independence of \mathcal{G} on \dot{x} is all that is needed to conclude the Lorentz force law¹¹ for Yang-Mills theory:

$$\mathcal{F}^\mu(x, \dot{x}, I) = \mathcal{G}^\mu(x, I) + F^{\mu\nu}(x, I)\dot{x}_\nu. \quad (30)$$

The very same methods used above can be used to show

$$d * \mathcal{G}(x, I) = 0, \quad (31)$$

the Bianchi identity for $\mathcal{G}^\mu(x)$, where the derivative is the usual exterior one and the Hodge star is used.

Finally we wish to demonstrate that we are really doing Yang-Mills gauge theory by defining the gauge transformation of the Yang-Mills gauge potential. We define gauge transformations in a novel way and then show that the resulting formalism and conceptual content is the same as the one inherent in the fiber bundle interpretation of Yang-Mills gauge theory [29].

In order to motivate the construction, consider that what we must express in mathematical terms is the symmetry of the physics under a certain group G whose algebra is \mathfrak{g} . The physics is described by \mathfrak{I} and the analytic conditions described in the axioms, so we first have to represent the action of G on \mathfrak{I} ; however, we can be more specific, because the Yang-Mills gauge symmetry leaves the space-time or external structures (sometimes called ‘horizontal’ in the geometric language) invariant by fiat (this is the *verticality* requirement in the fiber bundle language), so we expect that that action should only act non-trivially on \mathfrak{g} . The relevant mathematical construction should now begin to clarify, at least insofar as it must involve the adjoint action.

To review, since every Lie algebra \mathfrak{f} is a vector space via the forgetful functor which forgets the Lie bracket, one can consider using this vector space itself as the representation space for a representation $\check{\Gamma}(F)$ of the group F . This can be accomplished by applying the exponential map to the adjoint representation $\check{\Gamma}(\mathfrak{f})$ of \mathfrak{f} . Since the Yang-Mills gauge potential, $A_\mu(x, I^a)$ is introduced as a function on \mathfrak{I} it will transform as well—this will define a gauge transformation.

Let $\Lambda \in \mathfrak{g}$. Then

$$\delta I^a = [\Lambda, I^a] \quad (32)$$

¹⁰Further details can be found in [22].

¹¹Modulo the scalar field contribution $\mathcal{G}^\mu(x)$.

is the variation of the basis element I^a under Λ . This can be further expanded

$$\delta I^a = [\Lambda_c I^c, I^a] \quad (33)$$

$$= \Lambda_c(x) [I^c, I^a] \quad (34)$$

$$= \Lambda_c(x) f_d^{ca} I^d \quad (35)$$

$$= \check{\Lambda}_d^a(x) I^d \quad (36)$$

$$=: \check{\Lambda}^a(x). \quad (37)$$

Therefore,

$$\delta \check{A}^{\mu a} = \frac{m}{i\hbar} [\dot{x}^\mu, \delta I^a] \quad (38)$$

$$= \frac{m}{i\hbar} [\dot{x}^\mu, \check{\Lambda}^a]. \quad (39)$$

On the other hand, we have to take into account the fact that $\check{A}^{\mu a}$ is related to $A_\mu(x)$ by (13), and it is $\delta A_\mu(x)$ that we wish to compare with the standard result.

$$\delta \check{A}^{\mu a} = \delta((\check{A}^\mu)_b^a I^b) \quad (40)$$

$$= \delta(\check{A}^\mu)_b^a I^b + (\check{A}^\mu)_b^a \delta I^b \quad (41)$$

$$= \delta(\check{A}^\mu)_b^a I^b + (\check{A}^\mu)_b^a \check{\Lambda}^b \quad (42)$$

implies

$$\delta(\check{A}^\mu)_b^a I^b = \frac{m}{i\hbar} [\dot{x}^\mu, \check{\Lambda}^a] - (\check{A}^\mu)_b^a \check{\Lambda}^b. \quad (43)$$

A little computation shows that the first term on the RHS of (43) can be written

$$\frac{m}{i\hbar} [\dot{x}^\mu, \check{\Lambda}^a] = \check{\Lambda}_b^a \check{A}^{\mu b} + \partial^\mu \check{\Lambda}_b^a I^b. \quad (44)$$

Substituting (44) into (43) we get

$$\delta(\check{A}^\mu)_b^a I^b = \check{\Lambda}_b^a \check{A}^{\mu b} + \partial^\mu \check{\Lambda}_b^a I^b - (\check{A}^\mu)_b^a \check{\Lambda}^b. \quad (45)$$

The first and last terms on the RHS can be combined into

$$[\check{\Lambda}, \check{A}^\mu]_b^a I^b \quad (46)$$

by using the definitions of $\check{A}^{\mu b}$ and $\check{\Lambda}^b$. The final result is

$$\delta(\check{A}^\mu)_b^a = [\check{\Lambda}, \check{A}^\mu]_b^a + \partial^\mu \check{\Lambda}_b^a, \quad (47)$$

which is in agreement with the standard gauge transformation derived from the geometric fiber bundle approach.

Having established that the infinitesimal gauge transformation of the Yang-Mills gauge field introduced in (12) is formally identical to the one introduced by Yang and Mills, one can ask whether the equations really have the same

interpretation. The infinitesimal space-time translation of a function on \mathfrak{I} is generated by \dot{x}^μ and therefore we can *reinterpret* (12) as an infinitesimal space-time translation of the color vector I^a :

$$I^a \xrightarrow{\text{trans.}} \check{A}_b^a(x) I^b \quad (48)$$

$$[\dot{x}, I^a] = [\dot{x}^\mu, I^a] \partial_\mu = \check{A}^a = \check{A}_b^a I^b \quad (49)$$

In other words the gauge potential determines the behavior of the color vector under an infinitesimal translation in space-time—the translated vector is simply rotated by the gauge field in the adjoint representation. This provides a meaning of parallelism in the geometric picture, and in this algebraic picture.

3 Relaxing the condition of verticality and generalizing the concept of local symmetry

In the prior section we have assumed that \mathfrak{g} is the direct sum of commuting compact simple and $U(1)$ subalgebras and associated strictly with an ‘internal’ space. Here we relax those conditions and allow the gauge algebra \mathfrak{g} to act non-trivially on the space-time variables in \mathfrak{I} , namely $\{x^\mu, \dot{x}^\mu\}$. In particular we introduce an overlap between the “algebra” of displacements and the gauge algebra. We put quotes around algebra because once we introduce the generators of local displacements, we will see that the resulting structure is more general than a Lie algebra, it is call a Lie algebroid. We will not need any deep results from Lie algebroid theory in this work, but future work will emphasize this aspect—it is the key new idea in our formulation of a generalized gauge theory which incorporates gravity. For the purposes of this note simply imagine the notion of local symmetry extended such that the structure constants that determine the traditional symmetry algebra are allowed to vary from point to point. The varying structure constants in the momentum algebra will turn out to be the gravitational field. Dynamics for this structure will be introduced in later work.

The first attempt to understand gravity in the spirit of the fiber bundle language was by Utiyama [25], who took the gauge algebra to be $\mathfrak{so}(3, 1)$; Utiyama’s intuition was that behavior associated to gravitation would arise from gauging the the algebra $\mathfrak{so}(3, 1)$. This is a natural expectation for a particle physicist since locally, where there is a negligible space-time deformation, a great deal of the structure of physical laws is governed by Lorentz invariance. Interestingly however, Utiyama had to strategically insert *tetrads* in an ad hoc way in order to make the construction consistent, and we will see why. Since that attempt it has become quite clear, starting with [14] that (at least some) behavior we think of as peculiarly gravitational is induced by local translations, not by local Lorentz rotations. This section will provide a simple proof of this fact, not by analyzing the dynamics of the gravitational fields directly, but by considering the *motion of test particles in an ambient gravitational field*, and then showing

how the introduction of local translation symmetry introduces the correct equations of motion and generates the relevant gravitational potentials appearing in them.

3.1 Local Lorentz symmetry

Here we take $\mathfrak{g} = \mathfrak{so}(3, 1)$ following [25] thereby obtaining for \mathfrak{I} ¹²

$$[x^\mu, x^\nu] = 0 \quad (50)$$

$$[x^\mu, \dot{x}^\nu] = \frac{-i\hbar}{m} \eta^{\mu\nu} \quad (51)$$

$$[S^{ab}, S^{cd}] = f^{abcd}_{\quad\quad\quad ik} S^{ik} \quad (52)$$

$$[x^\mu, S^{ab}] = 0 \quad (53)$$

where the adjoint representation and the structure constants of the Lorentz algebra are defined as

$$(\check{S}^{cd})^{ab}_{ik} := f^{abcd}_{\quad\quad\quad ik} := -i(\eta^{ac}\delta^d_i\delta^b_k - \eta^{bc}\delta^d_i\delta^a_k + \eta^{ad}\delta^c_i\delta^b_k - \eta^{bd}\delta^c_i\delta^a_k). \quad (54)$$

The Lorentz potentials are introduced as

$$(\check{A}^\mu)^{ab}(x, \dot{x}, S) := m[x^\mu, \dot{S}^{ab}] \quad (55)$$

with the same conventions as in the Yang-Mills case. The analogue of the Wong equations is

$$\dot{S}^{ab} - m(\check{A}_\mu)^{ab}_{\quad cd}\dot{x}^\mu S^{cd} = 0 \quad (56)$$

which is the statement that the tensor $S^{ab}(\tau)$ is parallel transported along the curve associated with the parameter of differentiation, with respect to the Lorentz connection $(\check{A}_\mu)^{ab}_{\quad cd}$. Following this path leads in the end to a Lorentz force law of the form

$$\mathcal{F}^\mu(x, \dot{x}, S) = \mathcal{G}^\mu(x, S) + F^{\mu\nu}_{\quad ab}(x)\dot{x}_\nu S^{ab}, \quad (57)$$

where \mathcal{G} is linear in S^{ab} . The important characteristic of (57) is its linearity in the velocity. By going through the derivation, it is rather difficult to imagine how that problem is to be solved, even by the addition of the translation generator for the Poincaré group. The origin of quadratic force law is quite mysterious at this point, but it is also very necessary, in fact it has been argued [19] that certain fundamental aspects of the gravitational field equations can be derived from the force law for test particles, and that the term going as the square of the velocity is essential in this regard. Certainly the reciprocal problem of determining the

¹²The reason $[x^\mu, S^{ab}] = 0$ is that x^μ are coordinates and do not represent the components of the position vector. A vector character may be attributed to x^μ in flat space, but must be abandoned in the manifold model for space-time. When an affine structure is granted to the tangent space, something called a Cartan radius vector can be introduced as a sort of local position vector—this will be introduced in later work and the issue will be revisited as to what to take for the commutator between the Cartan vector and the Lorentz generators.

law of motion from the field equations comprises a set of rigorous results falling under the rubric, ‘*the problem of motion*’. In the problem of motion studies it is clear that the hallmark of distinctly gravitational equations of motion is their dependence on the *square* of the velocity of the test particle.

3.2 Direct products

We take this opportunity to sketch the results for direct products demonstrating that the problem will persist through their introduction. The purpose of this exercise is to leave one fewer in a short list of possible solutions to the quadratic velocity problem and to pursue it for its intrinsic interest.

Suppose we consider $\mathfrak{g} = \mathfrak{g}_1 \otimes \mathfrak{g}_2$ with basis $\{I_i^a \otimes I_j^b\}$. We make all the usual assumption, e.g. linearity of all functions in the generators, and introduce a pair of gauge potentials $\{(\tilde{A}_\mu^{I_i})_b^a(x)\}_{i=1}^2$, where $\{I_1^a\} = \{I^a\}$ and $\{I_2^a\} = \{J^a\}$ in the usual way and the result is a direct product of the construction we have already presented. In particular, the Lorentz force law is

$$\mathcal{F}^\mu(x, \dot{x}, I_i) = \mathcal{G}^\mu(x, I_i) + \sum_i F_i^{\mu\nu}{}_a(x) \dot{x}_\nu I_i^a. \quad (58)$$

The basic characteristics of this result would not be affected by further products.

Thus we seem to have only a few options left

- The quadratic law follows from the fact that the Poincaré group is the *semi*-direct product of the translation and Lorentz groups. In other words the dynamical structure of the geodesic type equation stems from the particular *way in which the translations are algebraically treated* in the Poincaré group.
- The field strengths and potentials are not linear in the gauge generators. This would change the whole structure of gauge theory and might be very interesting to study.
- There is *something unusual about the translation generators* that will somehow fix the problem. If this is true we might consider dispensing with the Lorentz generators altogether and seeing if gravitational behavior remains—we will do this.

The third possibility is a genuine solution—the second will not be explored here. This was gradually understood after the Utiyama paper. Kibble [14], it is often said, treated translations *on a par* with Lorentz transformations by using the entire Poincaré group. While this is true in the sense that he did not introduce the tetrads in an ad hoc way—they are dynamical variables and not background structures—the significant difference between the behavior of the tetrads and the spin connection is not emphasized by the language “on a par with”.

Our view is that the difference is substantial and has been underplayed. Statements in the literature that general relativity is a special case of gauge theory in which the fiber is the tangent space are very common and we think

incorrect. It is an active topic of research [24, 23] to find a proper mathematical setting for a gauge theory involving external (space-time) symmetries.

It should be mentioned that possibilities one and three posses some overlap—there is no analogue of the Poincaré group in which translations are combined with rotations via direct products, however some peculiarly gravitational behavior still cannot be associated with the *way in which the translations are embedded* into the Poincaré group due to the remarkable fact that the Lorentz part of the Poincaré group can be *ignored* for the purpose of constructing the equation of motion for spinless test particles—therefore the analysis need not be modified.

3.3 Local translation symmetry

The basic insight that allows the introduction of local translation invariance, and consequently gravity into this algebraic framework is contained in [12], thus we review their argument. It is couched in a geometric language, but it will be a simple matter to transplant the ideas into our more general algebraic framework.¹³

Consider the space of 1-forms with basis $\{dx^i\}$ on a manifold \mathcal{M} . The $\{dx^i\}$ are globally translation invariant, but not locally so:

$$x^i \mapsto x'^i = x^i + \epsilon^i \quad \epsilon^i \in \mathbb{R} \quad (59)$$

implies

$$dx^i \mapsto dx'^i = dx^i, \quad (60)$$

whereas

$$x^i \mapsto x'^i = x^i + \epsilon^i(x) \quad \epsilon^i(x) \in \mathcal{C}^\infty(\mathbb{R}^4) \quad (61)$$

implies

$$dx^i \mapsto dx'^i = dx^i + d\epsilon^i. \quad (62)$$

In this circumstance we can introduce a compensating field $\overset{t}{A}{}^a(x)$ (t for translations), which will allow us to make a basis of differential forms $\{Dx^i\}$ that is invariant under local translations, provided the potential transforms properly. Specifically,

$$\delta \overset{t}{A}{}^a = -\delta^a_i d\epsilon^i \quad \Rightarrow \quad \delta Dx^i = 0. \quad (63)$$

and

$$e^a = \delta^a_i Dx^i = \delta^a_i dx^i + \overset{t}{A}{}^a \quad \Rightarrow \quad \delta e^a = 0 \quad (64)$$

This is simply an alternative to the usual way of introducing the tetrad basis that emphasizes its origin as a translational potential, in accordance with the findings of [14]. Note that it is not quite the local translation potential, but

¹³With the advent of noncommutative geometry, and the unification of mathematics that it entails, it is beginning to be awkward to make such distinctions in the first place.

it *is* an invariant function of it. Actually the correct identification of the local translation potential has been a controversial issue. The difficulty is that the tetrad does not have the correct transformation properties, for a review see [23].

In terms of the elements of \mathfrak{I} we might expect the following associations (\leftrightarrow designates an association)

$$\delta^a_i dx^i \leftrightarrow \delta^a_\mu \dot{x}^\mu \quad (65)$$

and

$$\delta^a_i dx^i + A^a \leftrightarrow \delta^a_\mu \dot{x}^\mu + \overset{t}{A}^a \quad (66)$$

where

$$\overset{t}{A}^a = \overset{t}{A}^a_\mu \dot{x}^\mu \quad (67)$$

on the right hand side of (66). $\overset{t}{A}^a(x)$ may be thought of as some kind of gauge field fluctuation, similar in type to the ones encountered when constructing invariant coordinates in noncommutative theories, see e.g. [8] p. 984. We can now finally define a new element of \mathfrak{I}

$$e^a(x, \dot{x}) := \delta^a_\mu \dot{x}^\mu + \overset{t}{A}^a_\mu(x) \dot{x}^\mu. \quad (68)$$

Here no computation needs to be done to ensure that there is no velocity dependence in $\overset{t}{A}^a_\mu(x)$. It is important to notice that the temptation to use the name \dot{y}^a for e^a leads to a mathematical contradiction as the fluctuation is generally non-holonomic; there is no coordinate whose derivative gives the right hand side of (68).

We now extend the algebra \mathfrak{I} by replacing \dot{x}^μ by (68). Making the definitions

$$-i\hbar W_{\mathfrak{I}}^{ab} := m^2 [e^a, e^b] = [\mathcal{P}^a, \mathcal{P}^b] \quad (69)$$

$$-(\frac{m^2}{i\hbar}) h^a_\mu h^b_\nu [\dot{x}^\mu, \dot{x}^\nu] = h^a_\mu h^b_\nu W_{\mathfrak{g}}^{\mu\nu} =: W_{\mathfrak{g}}^{ab}, \quad (70)$$

we can determine the relationship between the various field strengths associated with the algebras \mathfrak{g} , \mathfrak{I} , and the algebra associated with the local translations \mathfrak{b} .¹⁴ The *translation field strength* or *torsion*¹⁵ can be simply defined as the F-bracket of the translation covariant derivatives yielding

$$F^{ab} := D^a \overset{t}{A}^b - D^b \overset{t}{A}^a, \quad (71)$$

where $D^a = h^a_\nu \partial^\nu$. The \mathfrak{g} -field strength is the sum of any field strengths that enter into the theory via a potential introduced in the manner of 2 or 3.1—we can not tell what gauge potentials it is associated with by looking at (69)—we

¹⁴The reason we label the field strength associated with \mathfrak{g} using the letter W and not F is explained in the appendix A.

¹⁵Sometimes the word torsion is reserved for the translation field strength when all the indices have been translated to Greek with the aid of the tetrad components h^a_μ —by torsion we mean any form of the translation field strength.

also have to know what has been taken for \mathfrak{g} . Note that it is *not* associated only with internal symmetries since the field strength for Lorentz rotations are also contained in $F_{\mathfrak{g}}^{ab}$. All this may be suggestive of abelian gauge theory, but this is an illusion. In fact this is a remarkable situation in which the usual non-abelian commutator term has exactly the same form as the usual abelian exterior derivative term. It is essential to note that, since D is dependent upon $\overset{t}{A}$, the field strength is non-linear

$$\overset{t}{F}{}^{ab}(A_1 + \overset{t}{A}_2) \neq \overset{t}{F}{}^{ab}(A_1) + \overset{t}{F}{}^{ab}(A_2). \quad (72)$$

Of course this is the characteristic feature that determines the difference in the physics of abelian and non-abelian gauge theories. Physically it corresponds to the fact that the gauge particles carry the charge of the gauge group, in the case of gravity this is assured, as the alternative is the existence of an energy-less gauge boson. Thus from both the physical and mathematical points of view it is quite clear that, if gravity is to be thought of as a gauge theory, that gauge theory must have the behavior of a non-abelian gauge theory. We are explicit about this point due to assertions to the contrary in the literature.

3.4 The teleparallel equivalents of general relativity

We attribute the contentiousness of some discussions involving the geometric torsion to an unconscious continuation of the debate about the new quantum mechanics. Of course, quantum mechanics is not new, but when it was, it was more widely thought to be capable of, or in need of, an explanation in terms of classical concepts (see e.g. [10, 9] [4]) than it is today (see e.g. [20, 21]). Some physicists including Einstein, Schrödinger, and Weyl worked on unified field theories with an eye to eventually replacing the explanation of the stability of matter in terms of quantum theory by one in terms of a unified field theory that would rely upon an ontology more familiar to the western philosophical tradition, within which scientific progress has achieved great heights. These efforts at unified field theories widely utilized the torsion, and are widely considered an embarrassment. This phenomena should not, of course, be confused with differing judgements regarding the possible appearance of this tensor in laws more fundamental than general relativity or the standard model; neither should it be confused with differing judgements about the future of model building with torsion in various branches of physics from elasticity theory to quantum gravity—the latter are scientific questions and accordingly are to be probed by experiment and self-consistency analysis.

The usual metric formulation of general relativity has a wide number of classically equivalent¹⁶ formulations (and physically nontrivial extensions), one of which has come to be called *the teleparallel equivalent of general relativity*. This

¹⁶Up to boundary terms, which *are* important but do not affect the classical equations of motion, and thus, insofar as the motion of test particles follows from the field equations, does not influence them either.

latter is one type of teleparallel gravity theory—a theory of gravity in which a notion of distant parallelism is present *ab initio*. This is made possible due to the vanishing of the Cartan curvature throughout space-time. It is necessary to make substantial conceptual shifts when passing from general relativity to its teleparallel equivalent, but the only shift that is immediately necessary for the understanding of the present work is to cease thinking of gravity as a manifestation of the curvature of space-time and think of gravity as a force again, in consonance with the Newtonian view [17].

Another interpretation of the torsion tensor is provided in the Einstein-Cartan theory where the spin density of matter is viewed as the source of the torsion field [7]. We agree with the criticisms pointed out by Kleinert [15, 16] on the Einstein-Cartan theory and will not consider it any further.

There is however one important difference between the Newtonian gravitational force view and the teleparallel gravitational force view, in particular, the force of gravity in Newton's theory is not related to the gauging of a space-time symmetry, while in the teleparallel view it *is*—it will be a long known property of manifolds, namely nontrivial metric and torsion, as opposed to non-trivial metric and curvature that will be considered the origin of gravity. Therefore, in the teleparallel equivalent one expects there to be a gravitational Lorentz force law that governs the motion of test particles immersed in an ambient gravitational field. For our purposes we will not need the kinetic terms for the gravitational field itself, but will only require the analogue of the Lorentz force. First we display the result [1]:

$$h^a_\mu \frac{du_a}{ds} = F^a_{\mu\nu} u_a u^\nu. \quad (73)$$

$F^a_{\mu\nu}$ is the field strength for local translations, i.e. the torsion, and of course (73) is quadratic in the velocity as is appropriate for a gravitational force law. The $h^a_\mu(x)$ are the (non-holonomic) tetrad components, that can be broken up (non-canonically) into a holonomic and non-holonomic part.

First we will add the *generalized momenta*,

$$\mathcal{P}^a(x, p) := m e^a(x, \dot{x}) = m h^a_\mu(x) \dot{x}^\mu \quad (74)$$

to the objects in \mathfrak{I} , of course it is only a scalar multiple of the tetrad element written in the velocity basis. Direct computation gives

$$[x^\mu, \mathcal{P}^a(x, p)] = (-i\hbar) h^{a\mu}(x) \quad (75)$$

with the consistency condition that

$$h^{\mu a} = g^{\mu\nu}(x) h_\nu^a \quad (76)$$

i.e. that the metric is *not* trivial. Recall that, in the geometric approach to teleparallel gravity, the relevant geometry is the Weitzenböck one, which has nontrivial metric and torsion, as opposed to the Riemann one, which has nontrivial metric and curvature—thus we expect a non-trivial metric to present

itself. This arises via a deformation of the relation $[x, p] \sim \eta$. More precisely, we proceed by *assuming* (75) and (76) and *calculating* $[x, p]$:

$$h_\nu{}^a [x^\mu, p^\nu] = (-i\hbar) h^{\mu a} \Rightarrow [x^\mu, p^\nu] = (-i\hbar) g^{\mu\nu}(x). \quad (77)$$

In this approach to gravity, it is clear that, if it is going to work at all, it must be a very special or ‘degenerate’ case of gauge theory in one way and a very significant generalization in another. The way in which it is special manifests itself, in our approach, by the fact that the Wong equations for the local translation generators $\mathcal{P}^a(x, p)$, and the Lorentz force law for p^μ (the Heisenberg equations of motion), must *coincide*. The notion that gravitation is a *special* case of gauge theory was considered by Yang in 1974 [29, 30], among others, and has been codified in the notion that gravity is the gauge theory for which the fiber bundle is chosen to be the tangent bundle. That particular coincidence of fiber and tangent space, is represented, in our approach, by the identity of the Wong and Lorentz equations.

The way in which gravitation is a significant *generalization* of gauge theory is much more poorly understood, and it corresponds, in our approach, to the fact that the structure constant are actually *functions* on the manifold

$$[\mathcal{P}^a(x), \mathcal{P}^b(x)] = T^{ab}_c(x) \mathcal{P}^c(x). \quad (78)$$

This can be understood as an expansion of the meaning of *local symmetry* that is *different* from the one that was originally considered by Weyl. This kind of localization of symmetry can be considered to generalize the assumption that the symmetry of the theory, defined by the structure constants of a Lie algebra, cannot vary from point to point. This is presently being examined elsewhere and we do not need further results for the purposes of this work.

Since the $\mathcal{P}^a(x)$ satisfy the Jacobi identity, the Bianchi identity for the torsion follows:

$$(D^a T^{bc}_d + T^{bc}_e T^{ae}_d) + \text{cyclic} = 0, \quad (79)$$

where $D^a = h^{a\mu} \partial_\mu$.

4 Conclusions

We claim that when the pioneering ideas of Utiyama et. al. are pushed forward and enriched with a new principle of local symmetry a relatively clear theory of gauge gravity begins to emerge, the theory appears very friendly to traditional concepts of quantization due to its gauge flavor. We calculate the Wong equation of this theory. The Wong equation for a gauge theory describes the time evolution of the gauge charges associated with the particles in the theory. In the case of a translation gauge theory, the charges are the ‘internal momenta’, or ‘translation invariant momenta’ and the Wong equation produces an analogue to the electromagnetic Lorentz force law for the gravitational case. This equation is familiar from the theory of teleparallel gravity, which is a theory exactly equivalent to Einstein’s general relativity, and this equation corresponds to the

geodesic equation. There are a few obvious next steps on this direction, one surprising feature that the newly found Lagrangian formulation for the Wong equations, when applied to the equation for the gauge invariant momentum, requires the introduction of Grassmann variables.

A Proof of the gravitational analogue of the Lorentz force law

Taking the time derivative of $[x^\mu, \mathcal{P}_a(x, \dot{x})] = -i\hbar h_a^\mu(x)$ gives

$$[\dot{x}^\mu, \mathcal{P}_a] + [x^\mu, \dot{\mathcal{P}}_a] = -i\hbar \partial^\epsilon h_a^\mu \dot{x}_\epsilon; \quad (80)$$

while multiplying this by h_λ^a yields

$$h_\lambda^a [\dot{x}^\mu, \mathcal{P}_a] + [x^\mu, h_\lambda^a \dot{\mathcal{P}}_a] = -i\hbar h_\lambda^a \partial^\epsilon h_a^\mu \dot{x}_\epsilon. \quad (81)$$

The first term on the LHS can be written

$$h_\lambda^a [\dot{x}^\mu, \mathcal{P}_a] = (i\hbar) (h_\lambda^a \partial^\mu h_{a\epsilon} \dot{x}^\epsilon - \frac{1}{m} g_{\lambda\epsilon} W_{\mathfrak{g}}^{\mu\epsilon}) \quad (82)$$

Following Feynman's logic we compute the equation of motion by noting that

$$[x^\mu, h_\lambda^a \dot{\mathcal{P}}_a] = -(\frac{i\hbar}{m}) \partial_x^\mu (h_\lambda^a \dot{\mathcal{P}}_a), \quad (83)$$

and integrating the resulting equation with respect to the velocity. When we do this, we will get teleparallel gravity's equation of motion for test particles in an ambient gravitational field, which is simply another form of the geodesic equation.¹⁷

One might be tempted to suppose that the only difference between the Yang-Mills and translation cases is the presence of x dependence on the right hand side of (81), however, there is a second subtlety, nicely treated in [22], which explains the choice of the label $W_{\mathfrak{g}}$, as opposed to $F_{\mathfrak{g}}$, for the F-bracket of velocities. The point is that the F-bracket of velocities, which we call $W_{\mathfrak{g}}$, is *itself* velocity dependent, albeit only linearly, whereas the combination

$$F_{\mathfrak{g}}^{\mu\nu}(x) := W_{\mathfrak{g}}^{\mu\nu}(x, \dot{x}) - m(\partial^\nu g^{\lambda\mu} - \partial^\mu g^{\lambda\nu}) \dot{x}_\lambda \quad (84)$$

is independent of the velocity. Naturally we reserve the label $F_{\mathfrak{g}}$ for the (velocity independent) field strength associated with the algebra \mathfrak{g} .

After expanding the derivatives of the metric, we rewrite this using the Weitzenböck connection

$$\Gamma_{\mu\nu}^\lambda := h_a^\lambda \partial_\nu h_a^\mu, \quad (85)$$

$$\begin{aligned} W_{\mathfrak{g}}^{\lambda\sigma} &= F_{\mathfrak{g}}^{\lambda\sigma} - mg^{\sigma\delta} g^{\epsilon\eta} \Gamma_{\eta\delta}^\lambda \dot{x}_\epsilon - mg^{\sigma\delta} g^{\lambda\eta} \Gamma_{\eta\delta}^\epsilon \dot{x}_\epsilon \\ &\quad + mg^{\lambda\delta} g^{\epsilon\eta} \Gamma_{\eta\delta}^\sigma \dot{x}_\epsilon + mg^{\lambda\delta} g^{\sigma\eta} \Gamma_{\eta\delta}^\epsilon \dot{x}_\epsilon. \end{aligned} \quad (86)$$

The remaining $h\partial h$ terms in (81) and (82) can be written in terms of connections as

$$h_\lambda^a \partial^\epsilon h_a^\mu = -g^{\epsilon\sigma} \Gamma_{\lambda\sigma}^\mu \quad (87)$$

$$h_\lambda^a \partial^\mu h_{a\epsilon} = g_{\lambda\eta} g^{\mu\delta} \Gamma_{\epsilon\delta}^\eta. \quad (88)$$

¹⁷Keeping track of ordering yields some quantum corrections which we ignore

Using this along with (85), we find the relation

$$\begin{aligned}
-(\frac{i\hbar}{m})\partial_x^\mu(h^a_\lambda\dot{\mathcal{P}}_a) = & i\hbar g^{\epsilon\sigma}\Gamma^\mu_{\lambda\sigma}\dot{x}_\epsilon \\
& - i\hbar g_{\lambda\eta}g^{\mu\delta}\Gamma^\eta_{\epsilon\delta}\dot{x}^\epsilon \\
& + (\frac{i\hbar}{m})g_{\lambda\epsilon}\{F_g^{\mu\epsilon} \\
& - mg^{\epsilon\delta}g^{\rho\eta}\Gamma^\mu_{\eta\delta}\dot{x}_\rho \\
& - mg^{\epsilon\delta}g^{\mu\eta}\Gamma^\rho_{\eta\delta}\dot{x}_\rho \\
& + mg^{\mu\delta}g^{\rho\eta}\Gamma^\epsilon_{\eta\delta}\dot{x}_\rho \\
& + mg^{\mu\delta}g^{\epsilon\eta}\Gamma^\rho_{\eta\delta}\dot{x}_\rho\}.
\end{aligned} \tag{89}$$

The second and sixth term on the RHS cancel identically, while using the definition

$$T^\mu_{\nu\rho} := \Gamma^\mu_{\rho\nu} - \Gamma^\mu_{\nu\rho} \tag{90}$$

the first with fourth terms, and the second with the last term combines respectively to give two torsion terms and (89) becomes:

$$-\frac{1}{m}\partial_x^\mu(h^a_\lambda\dot{\mathcal{P}}_a) = g_{\lambda\epsilon}F_g^{\mu\epsilon} - g^{\rho\eta}T^\mu_{\lambda\eta}\dot{x}_\rho - g^{\mu\eta}T^\rho_{\lambda\eta}\dot{x}_\rho. \tag{91}$$

The integration of (91) respect to \dot{x}_μ gives

$$\frac{1}{m}h^a_\lambda\dot{\mathcal{P}}_a = T^\epsilon_{\lambda\delta}\dot{x}_\epsilon\dot{x}^\delta - F_g^{\mu\delta}\dot{x}_\mu\dot{x}^\delta, \tag{92}$$

and being the translation field strength:

$$F_g^t_{\mu\nu} = h^a_\lambda T^\lambda_{\mu\nu} \tag{93}$$

it follows that (92) is in accordance with (73).

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